# Coherence of Triangle Artin Groups

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**Abstract**

Artin groups have attracted a great deal of current research due to their strong connections to geometry. We investigate the relationship of Artin groups to 3-manifold groups and analyze the properties of their subgroups.

## 1 Introduction

A (mathematical) *knot* is essentially a tangled or knotted closed loop. More formally, a knot is an embedding of a circle $S^1$ in $\mathbb{R}^3$ or $S^3$. It is clear that there exist infinitely many embeddings for each knot. However, if two knots can be transformed into one another by deforming the ambient space, they are said to be *isotopic* and are actually considered to be the same knot.

The idea of a knot can then be extended to arrive at the definition for a link. A $k$-component *link* consists of $k$ knots which may or may not be intertangled, and a knot is therefore just a 1-component link. Since almost all of the definitions and theorems for knots can be extended to links, we will use the terms *knot* and *link* interchangeably from now on.

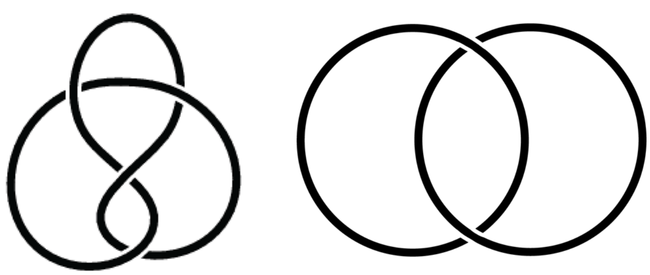

Figure 1: The figure eight knot and the Hopf link.

Knots are hard to work with in $\mathbb{R}^3$ or $S^3$ so mathematicians generally use knot projections which are also known as (planar) *knot diagrams*. Figure 1 depicts standard knot and link diagrams. At each crossing of a knot diagram, there can only be one under-strand and one over-strand. In particular, three or more strands cannot meet at a crossing - if such pathology occurs, strands should be shifted to make all the crossings legal. A resolution of a problematic projection is demonstrated in Figure 2.

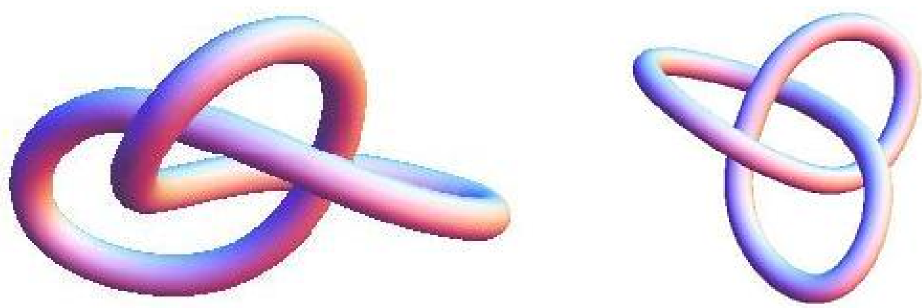

Figure 2: An illegal projection vs. a legal projection of the trefoil knot.

But how can we distinguish between isotopic and non-isotopic knots? This is the most fundamental question in knot theory. Though the problem may appear to be elementary, knot theory draws from deep and seemingly unrelated areas of mathematics to tackle this problem.

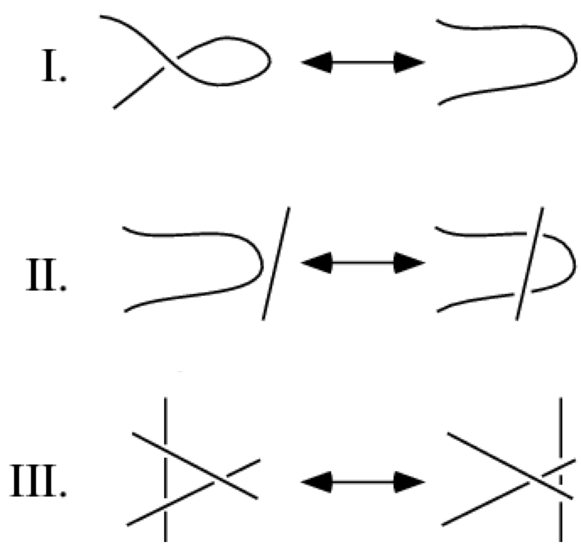


Figure 3: The three Reidemeister moves.

J.W. Alexander, G.B. Briggs, and Kurt Reidemeister proved that using only the three Reidemeister moves, any two knot diagrams of the same (isotopic) knot can be transformed into each other. Figure 3 illustrates these three moves. It might seem to be feasible to quickly determine the isotopy class of every knot using Reidemeister moves - we would just find a sequence of Reidemeister moves taking one knot projection to another. However, no bound has been found on the number of Reidemeister moves needed to transform one knot diagram to another. For instance, it may be necessary to take, using Reidemeister moves, a 10 crossing knot projection to more than one million crossings before it can be taken back down to a 9 crossing knot projection.

A *knot invariant* is a function which maps isotopic knots to the same "value." Functions of knot diagrams that are invariant under the Reidemeister moves are invariant under knot isotopy, so the validity of many invariants are proved by showing invariance under the Reidemeister moves. A large number of invariants have been found and the "values" can range anything from integers to groups. For instance, the *linking number* of a 2-component link is one of the most basic invariants.

**Definition 1.1.** Let $D$ be the link diagram of a 2-component link $L$. Using Figure 4, define $\epsilon(c) = \pm 1$ on a crossing $c$ of $D$. The *linking number* of $L$ is $\frac{1}{2}\sum \epsilon(c)$, summed over the crossings between the two different components.

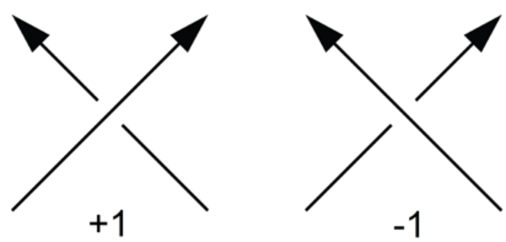


Figure 4: $\epsilon(c) = +1$ and $\epsilon(c) = -1$ respectively.

To show that the linking number is invariant under the Reidemeister moves is a relatively straightforward exercise. The linking number is also easy to understand and compute. Unfortunately, the invariant is quite weak, i.e. the linking number fails to distinguish very simple non-isotopic links. The Whitehead link shown in Figure 5 is such an example - the Whitehead link has the same linking number as the 2-component unlink. Also, the invariant is extremely limited as it can only be applied to 2-component links.

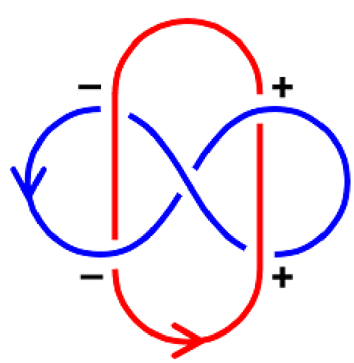


Figure 5: The Whitehead link with linking number = 0.

Stronger invariants like the Jones polynomial can distinguish far more knots. However, each invariant has an equally impressive set of knots that it cannot distinguish. In fact, it is still an open question on whether or not there exists a non-trivial knot that has the same Jones polynomial as the unknot.

# 2 Artin Groups

**Definition 2.1.** An *Artin* group is a group with generators $a_1, a_2, a_3, ..., a_n$ and relations of the form $\langle a_i a_j \rangle^{m_{ij}} = \langle a_j a_i \rangle^{m_{ij}}$ which denote

$$\underbrace{a_i a_j a_i \cdots a_j a_i}_{m_{ij}} = \underbrace{a_j a_i a_j \cdots a_i a_j}_{m_{ij}}.$$

Often, Artin groups are presented as a graph $\Gamma$ with labeled edges as drawn in Figure 6, and the Artin group associated to $\Gamma$ is denoted $A\Gamma$. An unlabeled edge indicates that $m_{ij} = 2$.

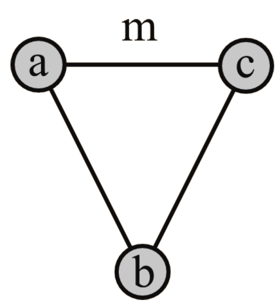


Figure 6: The Artin group $A(2, 2, m)$ with presentation $\langle a, b, c \mid ab = ba, bc = cb, \langle ac \rangle^m = \langle ca \rangle^m \rangle$.

**Definition 2.2.** The *Coxeter* group which corresponds to an Artin group adds the additional relations, $a_1^2 = a_2^2 = a_3^2 = \ldots = a_n^2 = 1$.

**Definition 2.3.** A *right-angled Artin* group is an Artin group with all $m_{ij} \in \{2, \infty\}$. Note that $m_{ij} = \infty$ means that there is no relation between the generators $a_i$ and $a_j$ so the relation is usually omitted.

Right-angled Artin groups encompass everything from a graph with no edges which represents a free group on $n$ generators to a complete graph which represents a free abelian group on $n$ generators.

**Definition 2.4.** A *3-manifold group* is a group that is isomorphic to the fundamental group of a connected 3-manifold.

In fact, the fundamental group of the knot complement $\pi_1(S^3 - K)$ is a 3-manifold group. Gordon and Luecke proved that the knot $K$ is completely determined by its knot complement [GL], so if two knots have homeomorphic complements, there is a homeomorphism of the 3-sphere transforming one knot into the other.

Droms first found that a right-angled Artin group is a 3-manifold group if and only if every component of the corresponding graph is a tree or a triangle. The triangle where each edge is labeled 2 is isomorphic to $\mathbb{Z}^3$. The triangle is also a 3-manifold group since $\mathbb{Z}^3 \cong \pi_1(T^3)$. Later, Hermiller and Meier showed that if $L$ is the connected sum of $(2, m_i)$ torus links, $\pi_1(S^3 - L)$ is isomorphic to the Artin group $A\Gamma$, where $\Gamma$ is a tree with labels $m_i$ [HM]. Gordon demonstrated that these are the only connected graphs $\Gamma$ whose Artin groups $A\Gamma$ are 3-manifold groups [GO].

**Definition 2.5.** A labeled graph $\Gamma$ is of *finite type* if the Coxeter group corresponding to an Artin group is finite. On the other hand, $\Gamma$ is of *infinite type* if the Coxeter group corresponding to an Artin group is infinite.

When we use the term *triangles*, it will be in reference to Coxeter groups corresponding to Artin groups with three generators and with $m_{ij} \geq 2$. We want to know which triangles are of infinite type and which are of finite type. It is perhaps more natural to think of triangles as reflection groups but we will also analyze them combinatorially in the following few paragraphs.

**Lemma 2.6.** The Coxeter group $\mathcal{C}$ corresponding to the Artin group $A(2,2,m)$ is isomorphic to $\mathbb{Z}_2 \times \mathrm{D}_m$ where $\mathrm{D}_m$ is the $m^{th}$ dihedral group.

*Proof.* $\mathcal{C}$ has generators $a, b, c$ and relations $ab = ba, bc = cb, \langle ac\rangle^m = \langle ca\rangle^m, a^2 = b^2 = c^2 = 1$. Notice that $b$ is in the center of $\mathcal{C}$. Also, $b$ is its own inverse so every word in $\mathcal{C}$ either starts with one $b$ or contains no $b$'s when all the $b$'s are shifted to the beginning of the word. $\mathcal{C}$ is therefore isomorphic to $\mathbb{Z}_2 \times \langle a, c \mid \langle ac\rangle^m = \langle ca\rangle^m, a^2 = c^2 = 1\rangle$.

Now let $ac$ and $a$ replace $a$ and $c$ as generators respectively. Then,

$$(ac)^m = \underbrace{acacac\cdots acac}_{2m} = \underbrace{caca\cdots ca}_{m}\underbrace{acac\cdots ac}_{m} = 1$$

and $a(ac)a = ca = (ac)^{-1}$. Finally note that $\mathrm{D}_m$ has presentation $\langle x, y \mid x^m = 1, y^2 = 1, yxy = x^{-1}\rangle$.

□

***Remark.*** $(3,3,3)$, $(2,4,4)$, *and* $(2,3,6)$ *are intuitively of infinite type.*

*If we can find distinct incompressible words of arbitrary length for each triangle, the triangle would necessarily be of infinite type.* $(abc)(abc)(abc)a\ldots$ *is intuitively incompressible because no relations of* $(3,3,3)$ *can be applied to this sequence that would immediately shorten the word.* $(cbca)(cbca)(cbca)c\ldots$ *is similarly incompressible for the triangle* $(2,4,4)$*. Finally, although the relation* $ab = ba$ *of* $(2,3,6)$ *can be applied to* *make the sequence $(cacab)(cacab)(cacab)c\ldots$ appear to be different, no relations can be immediately used to compress the sequence.*

**Theorem 2.7.** *The only triangles of finite type are $(2,2,m)$, $(2,3,3)$, $(2,3,4)$, and $(2,3,5)$.*

*Proof.* Consider the triangles geometrically as reflection groups. For the triangle $(p,q,r)$, the angle of reflections would be $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$ respectively. If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$, the reflection group is spherical hence finite. Otherwise, if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, the reflection group is Euclidean, and if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ the reflection group is hyperbolic. It can then be seen that the triangles $(2,2,m)$, $(2,3,3)$, $(2,3,4)$, and $(2,3,5)$ are finite and all other triangles are infinite.

□

# 3 Coherence

**Definition 3.1.** A group $G$ is *coherent* if every finitely generated subgroup of $G$ is finitely presented. If there exists a subgroup of $G$ which isn't finitely presented, we call $G$ *incoherent*.

**Theorem 3.2.** *The Artin group with presentation $\langle a,b,x,y \mid a \leftrightarrow x, a \leftrightarrow y, b \leftrightarrow x, b \leftrightarrow y\rangle$, which is isomorphic to $F_2 \times F_2$, is incoherent.*

*Proof.* We claim that $G = \langle u,b,y \mid u^{-m}bu^m \leftrightarrow u^{-n}yu^n\rangle$, where $u = ax$ and $m,n \in \mathbb{Z}$, is a subgroup of $F_2 \times F_2$ which isn't finitely presented.

The quickest proof of this fact uses the theory of the homology of groups. Because the relations lie in the commutator subgroup of the free group on $u$, $b$, and $y$, it is not hard to see that the second homology group $H_2(G)$ is not finitely generated. Therefore $G$ is not finitely presented.

□

Gersten proved the incoherence of $F_2 \times F_2$ using a similar method [GE].

**Definition 3.3.** A graph $\Gamma$ is *chordal* if every cycle greater than three contains a chord.

The following proof of the coherence of chordal right-angled Artin groups is due to Droms [DR].

Let $\Gamma$ be a finite graph and let $G\Gamma$ be the associated right-angled Artin group. Given an element $g \in G\Gamma$, with $g = x_1^{e_1}x_2^{e_2}\cdots x_k^{e_k}$, where each $x_i$ is a vertex of $\Gamma$, we define

$$|g| = e_1 + e_2 + \cdots + e_k.$$

$|g|$ is independent of the expression of $g$ as a product of powers of generators, since each relation has exponent sum 0. Let $K\Gamma = \{g \in G\Gamma : |g| = 0\}$. It is clear that $K\Gamma$ is a subgroup of $G\Gamma$.

If $U$ and $V$ are full subgraphs of $\Gamma$, with $\Gamma = U \cup V$ and $\gamma = U \cap V$, then $G\Gamma = GU *_{G\gamma} GV$, as follows easily by examining generators and relations. In particular, if $U \cap V$ is empty, then $G\Gamma = GU * GV$. Since free products of 3-manifold groups are 3-manifold groups [HE, Lemma 3.2], and free products of coherent groups are coherent [KS, Theorem 8], it will suffice to prove the coherence of chordal right-angled Artin groups for connected graphs.

**Proposition 3.4.** Let $\Gamma$ be a finite connected graph, and let $U$ and $V$ be full subgraphs of $\Gamma$ with $\Gamma = U \cup V$ and $\gamma = U \cap V$. Then

$$K\Gamma = KU *_{K\gamma} KV.$$

*Proof.* Since $G\Gamma = GU *_{G\gamma} GV$, [SE, Theorem 13] implies that $G\Gamma$ acts on a directed tree $Y$, whose vertices are the left cosets of the subgroups $GU$ and $GV$ in $G\Gamma$, and whose edges are the left cosets of $G\gamma$ in $G\Gamma$. Thus $K\Gamma$ acts on $Y$ also. In fact, $K\Gamma$ acts transitively on the edges of $Y$; to see this, let $w$ be any vertex of $\gamma$, and let $gG\gamma$ be any edge of $Y$. Then $w^{|g|}g^{-1} \in K\Gamma$, and $(w^{|g|}g^{-1})(gG\gamma) = w^{|g|}G\gamma = G\gamma$, so there is only one orbit of edges under the action of $K\Gamma$. Since the vertices $GU$ and $G\gamma$ lie in different orbits of the $K\Gamma$-action on $Y$, the quotient directed graph $Y/K\Gamma$ consists of two vertices joined by an edge, so again by [SE, Theorem 13],

$$K\Gamma = K\Gamma \cap GU *_{K\Gamma \cap G\gamma} K\Gamma \cap GV = KU *_{K\gamma} KV.$$

□

**Theorem 3.5.** *If $\Gamma$ is a finite graph, then the group $G\Gamma$ is coherent if and only if each circuit of $\Gamma$ of length greater than three has a chord.*

*Proof.* Suppose every circuit of $\Gamma$ of length greater than three has a chord. If $\Gamma$ is complete, then $G\Gamma$ is finitely generated free abelian, and so coherent. Otherwise, $\Gamma$ has a separating set $A$ of vertices which induces a complete subgraph of $\Gamma$ [LO, Problem 9.29b]. That is, there are proper full subgraphs of $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\langle A \rangle = \Gamma_1 \cap \Gamma_2$, and $\langle A \rangle$ is complete. Thus,

$$G\Gamma = G\Gamma_1 *_{G\langle A \rangle} G\Gamma_2.$$

Every circuit of either $\Gamma_1$ or $\Gamma_2$ of length greater than three has a chord, so by induction, $G\Gamma_1$ and $G\Gamma_2$ are coherent. $G\langle A \rangle$ is finitely generated free abelian, so by [KS, Theorem 8], $G\Gamma$ is also coherent.

Now suppose that the graph $\Gamma$ is a circuit of length greater than three and let $x$ and $y$ be two nonadjacent vertices of $\Gamma$. Then there are proper full subgraphs $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \langle x, y \rangle$, and $\Gamma_1$ and $\Gamma_2$ are trees. Thus,

$$K\Gamma = K\Gamma_1 *_{K\langle x,y\rangle} K\Gamma_2.$$

Each of $K\Gamma_1$ and $K\Gamma_2$ is a finitely generated free group, so $K\Gamma$ is finitely generated. $K\langle x,y\rangle$ is the normal closure in the free group $G\langle x,y\rangle$ of $x^{-1}y$, so $K\langle x,y\rangle$ is not finitely generated. By [BA], $K\Gamma$ is not finitely presented, so $G\Gamma$ is not coherent. It follows that if some circuit of $\Gamma$ of length greater than 3 has no chord, then $G\Gamma$ has a incoherent subgroup, and is thus itself not coherent.

$\square$

Hermiller and Meier first determined that if $\Gamma$ is a cycle of length greater than or equal to 4, then $A\Gamma$ is incoherent [HM]. Gordon then found all triangle Artin groups of infinite type to be incoherent [GO]. He also found the finite type Artin groups $A(2,3,3)$ and $A(2,3,4)$ to be incoherent by noticing that $A(3,3,3)$ embeds in $A(2,3,4)$ which in turn embeds in $A(2,3,3)$. $A(3,3,3)$ is incoherent because it is of infinite type; therefore, $A(2,3,3)$ and $A(2,3,4)$ are also incoherent. On the other hand, $A(2,2,m) \cong A(m) \times \mathbb{Z}$ is coherent since $A(m)$ is a 3-manifold group, and 3-manifold groups are coherent by [SC]. However, the coherence or incoherence of $A(2,3,5)$ could not be established.

Unfortunately, $F_2 \times F_2$ is not a subgroup of $A(2,3,5)$. Also, there is no clear embedding of $A(3,3,3)$ in $A(2,3,5)$ so the method used by Gordon to prove the incoherence $A(2,3,4)$ does not work in this case.

As pointed out by Gordon, if $A(2,3,5)$ is incoherent, there is a nice characterization of the coherence or incoherence of any Artin group $A\Gamma$. That is, the Artin group is coherent if and only if $\Gamma$ is chordal, every complete subgraph of $\Gamma$ with 3 or 4 vertices has at most one edge with label greater than 2, and $\Gamma$ has no full subgraph of the form shown in Figure 7. If $A(2,3,5)$ is coherent, the characterization would be more complicated.

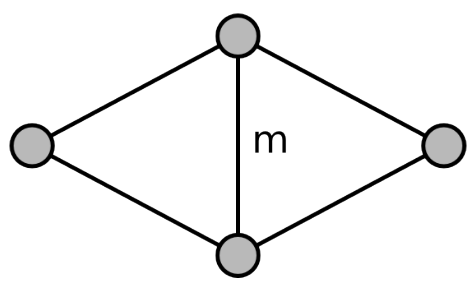


Figure 7: Subgraph where unlabeled edges have implicit label 2.

# 4 Acknowledgments

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